dent variable; $y = M_R/2.5M_\odot$ describes the scaling with total mass M_R; the subscript 0 refers to the $2.5M_\odot$ model; ν is the power of T in the energy generation law ($\nu = 4.5$ for p.p., $\nu = 18$ for CN used here). These relations use mass rather than radius as the independent variable. However the Tables I and II can be interpreted as giving all other variables as functions of $m = M/2.5M_\odot$. For different values of ν, the mass scaling changes according to Eq. (17) and the functions $r_0(m)$, $P_0(m)$, etc. need to be recalculated (note the difference between $\nu = 4.5$ and $\nu = 18$ models given here).

The homology relations exist because the opacity and energy generation laws are power law formulas. These power laws are only approximations. For lower mass main sequence stars, where the p.p. cycle provides most of the stellar luminosity, the p.p. cycle model given here, scaled to the mass of interest, gives a reasonable approximation. For higher mass main sequence stars' mass, where the CN cycle provides most of the luminosity, the scaled CN model gives a reasonable approximation.

In summary, following the prescription described in Secs. II and III above, it is possible to obtain relatively quickly self-consistent stellar models for stars with convective cores and radiative envelopes. The great simplification over the currently discussed methods in textbooks is that numerous trial core integrations need not be done. One only needs to solve Eq. (12) for ξ_c to get a core model that exactly matches r, P, and T at the envelope boundary. Equation (11c) gives the core mass and a single core integration then gives the core luminosity. The starting values of total mass and luminosity and surface radius for the envelope in general need to be adjusted a number of times to converge to a final model.

The above procedure makes it now practical to have upper year undergraduate astrophysics students construct self-consistent stellar models. This is much more satisfying to the student then constructing inconsistent models. It also demonstrates the Vogt–Russell theorem well. The values of total mass and luminosity and surface radius have to be chosen in a particular manner, consistent with the fact that a chemically homogeneous star's structure should depend only on its total mass and chemical composition.

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Generation and experimental measurement of a one-dimensional quasi-crystal diffraction pattern

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A microcomputer-generated one-dimensional quasi-crystal grating is produced on a high-resolution monitor. Photographic reproduction of the grating with about a 30× reduction in size produces an acceptable optical grating. The location of diffraction maxima and the relative intensity pattern produced by passing HeNe laser light through the grating is compared to the theoretical prediction with mixed results. The comparison of two types of film that can be used for this experiment is also discussed.

I. INTRODUCTION

In classical crystallography the atomic structure of a crystal is assumed to be periodic. A crystal thus consists of a single unit cell of atoms that is repeated throughout the crystal. As consequences of this translational symmetry, the possible rotational symmetries are limited to the 32 "crystallographic" point groups and the diffraction pattern of a crystal consists of sharp Bragg peaks. A remarkable discovery in 1984 by Shechtman *et al.*[1] showed a diffraction pattern of an alloy of aluminum and manganese with sharp diffraction peaks and a noncrystallographic point group symmetry. The sharp diffraction peaks implied long-range ordering. The noncrystallographic symmetry implied that the atomic structure was not periodic. This new type of structure, one that is not periodic yet gives rise to a diffraction pattern with sharp Bragg peaks is called a "quasi-crystal."

Microcomputers have been used for calculating theoretical diffraction patterns of the more complex Fresnel type,[2,3] and have been used to determine experimental intensities for diffraction and interference patterns.[4] In this article we describe a method of generating a one-dimensional quasi-crystal grating, obtaining a transparency image and investigating the diffraction pattern from that transparency. Briefly, the procedure is a one-dimensional quasi-crystal grating is generated on a computer screen and photographed to produce a transparency. A Frauenhofer diffraction pattern is produced by the transparency with a HeNe laser. The experimental intensity pattern is measured and compared to the predicted theoretical intensity pattern. Two types of film that could be used to produce the transparency are discussed.

The methods described may be applied to most college optics labs from the sophomore to senior level depending upon the degree of participation of the student in the various levels of preparation and investigation. It should also be noted that although we describe a quasi-crystal due to its current high degree of research interest, the methods discussed are very versatile, and may be used to produce a large variety of custom-designed optical "gratings" to be investigated.

II. THEORY

To generate quasi-crystal patterns (nonperiodic patterns that give rise to diffraction patterns with sharp Bragg peaks), so-called projection methods have been developed.[5-7] Projection methods are mathematical constructs that project sections of a hypercubic lattice onto lower dimensional spaces. The first such projection was given by de Bruijn, where he showed that the vertices of the two-dimensional Penrose pattern of darts and kites can be generated by the projection of a section of a five-dimensional hypercubic lattice into two dimensions.[5] One-dimensional quasi-crystal patterns are found by a projection from a two-dimensional square lattice into a one-dimensional space: On a two-dimensional square lattice with sides of unit length, one draws a line at an angle θ, with respect to one of the lattice directions, and displaced a distance d from an arbitrarily chosen origin. One then projects a corner, e.g., the lower left corner, of each square cut by the line, orthogonally onto the line. Equivalently, one can project all corners of the square lattice within a specified strip parallel to the line drawn.[8] The one-dimensional array of points on the line defines a one-dimensional quasi-crystal pattern. These one-dimensional quasi-crystal patterns consist of a row of points, each pair of consecutive points separated by one of two distinct segment lengths. These two segment lengths are referred to as long and short segments, and denoted by L and S, respectively. A characteristic of the quasi-crystal pattern is that the pattern of points or segment lengths is neither periodic nor random.

For these one-dimensional quasi-crystal patterns, an algebraic expression for the coordinates of the points has been derived.[9] This expression was derived using formulas for special sequences of the numbers one and zero given by de Bruijn.[10] Let $\lfloor x \rfloor$, called the floor of x, denote the largest integer less than or equal to x. Let $\lceil x \rceil$, called the roof of x, denote the smallest integer greater than or equal to x. The algebraic expression for the coordinates is given in terms of such floor and roof functions. Let $x(m)$ denote the coordinate of the mth point in a one-dimensional quasi-crystal constructed by the projection method with a line drawn on the two-dimensional square lattice at an angle θ and displacement d:

$$x(m) = m \cos\theta + (\lfloor \gamma + m/\alpha \rfloor - \lfloor \gamma \rfloor)(\sin\theta - \cos\theta), \tag{1}$$

where

$$\alpha = 1 + 1/\tan\theta,$$

$$\gamma = (-d/\sin\theta + \lceil -d/\sin\theta \rceil + 1)/(-\alpha).$$

The two segment lengths are, taking $45° \leqslant \theta \leqslant 90°$, $L = \sin\theta$ and $S = \cos\theta$.

Equation (1) can be generalized to include one-dimensional quasi-crystal patterns with two arbitrary distinct segments of length L and S:

$$x(m) = mS + (\lfloor \gamma + m/\alpha \rfloor - \lfloor \gamma \rfloor)(L - S). \tag{2}$$

This type of one-dimensional quasi-crystal pattern corresponds to those obtained via a second method known as the grid method in the case where one allows for arbitrary segment lengths.[11] Equation (2) has been utilized to construct diffraction gratings where the spacing between slits corresponds to a one-dimensional quasi-crystal pattern. In Fig. 1 we show such a grating whose spacings correspond to a Fibonacci quasi-crystal pattern: $d = 0$, $\theta = \arctan[(1 + \sqrt{5})/2]$, and $(1 + \sqrt{5})/2$ is the golden ratio.

The light intensity distribution for Frauenhofer diffraction from a quasi-crystal diffraction grating is calculated as follows: Let the incoming plane wave of wavelength λ (and wavenumber $k = 2\pi/\lambda$) be along the y axis normal to the diffraction grating in the x–z plane. The slits of width "a" are parallel to the z axis and the center of the slits is positioned at the x coordinates $x(m)$, $m = 1,2,3,\ldots,n$. The normalized intensity $I(\theta)$ of the beam diffracted at an angle θ in the x–y plane is given by

$$I(\theta) = (1/n^2)[F_c(\theta)^2 + F_s(\theta)^2]D(\theta), \tag{3}$$

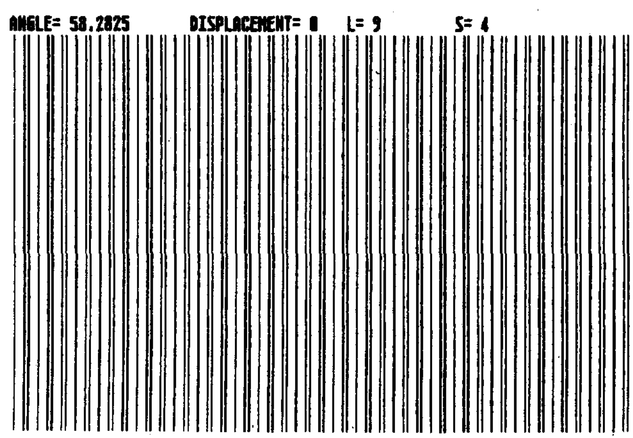

Fig. 1. A reproduction of the computer screen grating for a one-dimensional Fibonacci quasi-crystal. The screen is not color reversed.

where

$$F_c(\theta) = \sum_{m=1}^{n} \cos[kx(m)\sin\theta],$$

$$F_s(\theta) = \sum_{m=1}^{n} \sin[kx(m)\sin\theta],$$

and the diffraction factor is

$$D(\theta) = \sin^2[(\pi a/\lambda)\sin\theta]/[(\pi a/\lambda)\sin\theta]^2.$$

The intensity $I(x_s)$ as a function of the position x_s at a screen a distance d_s from the grating is found from the above equation by substituting $\sin\theta = x_s/d_s$.

III. EXPERIMENT

A computer program written in BASIC employing Eq. (2) was used to generate a vertical bar grid (grating) across 8 in. of the monitor of an Atari 520 ST computer.[12] This black and white grid was color reversed (Fig. 1 is the original "positive" computer screen) and reduced approximately 30× by photographing the screen from a distance of 1.5 m with a 35-mm camera equipped with a standard 50 mm, *f* 1.8 lens. The photographs were made using Polaroid HC Instant Copy Film. The resulting positive copy provided a "grating" approximately 7 mm across representing a one-dimensional Fibonacci quasi-crystal pattern.[13] The spacing and slit widths were originally determined by the computer program (limited by the resolution of the monitor screen), and the distance for photographing was determined such as to provide a reasonable diffraction pattern when illuminated with a HeNe laser of 5-mW intensity and 632.8-nm wavelength. The Polaroid film was chosen for its speed and convenience of development (2 min in a Polaroid Autoprocessor).

The developed exposures were cut apart, mounted in 2×2 slide mounts, and placed in an *x–y* translation stage located in front of the unmodified laser beam. The resulting diffraction pattern was projected onto a screen 4.6 m from the grating. The observed diffraction pattern was fit best with a computer model of 10 or 11 illuminated slits, and consistent with a determination of the $1/e^2$ beam radius (r_e) of 0.40 ± 0.04 mm from a knife-edge test. The center of the beam within $\pm r_e$ would illuminate about six slits of the grating. This illumination produced a finite number of maxima (about 13) and some visible secondary maxima.

The resulting experimental diffraction pattern is superimposed on the theoretical pattern in Fig. 2. The theoretical pattern is calculated from Eq. (3) and the experimental data were obtained using a photoresistor in a simple circuit (battery and series resistor). A digital voltmeter measured the potential across the series resistor. The potential was proportional to the intensity of the light falling on the photoresistor. The potential was corrected for the photoresistor response by calibrating it with a well-known double slit interference/diffraction intensity pattern using a technique similar to that in Ref. 4. The resulting intensity measurements are accurate to ±5%. The photoresistor was mounted on an optical bench in such a way that its horizontal position could be determined on the centimeter scale running the length of the bench.

An analysis of the data represented in Fig. 2 indicates the location of the diffraction maxima is a very good fit to theory for those maxima that are 4.86 cm or less from the central maxima (0.6 mm deviation for all but one maximum and 1.6 mm for that one), the maxima at 6.2 cm does not appear on the theoretical plot. It was discovered that the diffraction portion of Eq. (3), $D(\theta)$, has a minimum at 5.8 cm for the parameters that seem to provide the best fit to the rest of the pattern, and almost completely cancels the expected maximum. The intensity pattern for interference only [$D(\theta) = 1$ in Eq. (3)] does show the observed maximum at 6.2 cm. Examination of the computer screen image (Fig. 1) shows the slits (black lines) are not of constant width, but vary from one slit to another in spite of the programming calling for a slit of one pixel width in all cases. There appear to be two slit widths present. This phenomenon may explain the observed appearance of the maxima at 6.2 cm. We have investigated regular patterns such as a double slit with two widths and found similar anomalies of position and intensity.

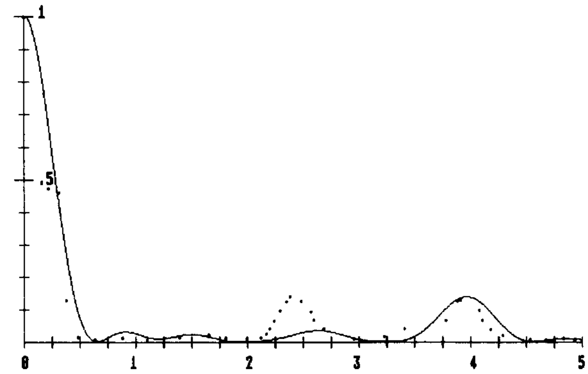

Fig. 2. A comparison of the normalized intensity pattern predicted by theory (solid line) and experimentally determined (dots). Only one half the diffraction pattern is shown because it is symmetric about the origin.

The intensity comparison is not as good. The measured intensities vary from 0.5 to 3 times the theoretical value. The theoretical intensity pattern was generated assuming a constant slit width. The observed variation from theory is of the kind associated with variable slit widths, such as we have seen in the double slit intensity pattern referred to above. The finite 8-mm aperture of the detector causes some loss of resolution of the detailed intensity pattern.

Although the Polaroid film is very convenient to use due to its speed and ease of processing, it has a serious drawback in the appearance of "grain" that does not yield a sharp edge at the slit boundary. At the end of this experiment, a comparison with Kodak Technical Pan Film that produces a fine grain negative when processed the more traditional way (D-19 for 6 min followed with 5 min in rapid fixer) was made. Figure 3 compares the results. Figure 3(a) and (b) provide a comparison of the negative images of a direct print from the slides with Fig. 3(a) being a portion of the Polaroid grating and Fig. 3(b) being the grating on Technical Pan film. The prints (about a 30× enlargement of the slides) show the difference in grain structure and the relative roughness of the slits on the Polaroid film due to the large grain present. Enlarged prints (about 3×) of 100× photomicrographs [Fig. 3(c) and (d)] emphasize the grain structure that appears in the lower magnification negative prints of Fig. 3(a) and (b). The comparable diffraction patterns are shown in Fig. 3(e) and (f), respectively. A significant amount of spurious diffraction is present in the pattern generated by the grating made from the Polaroid film. Surprisingly, however, the Kodak

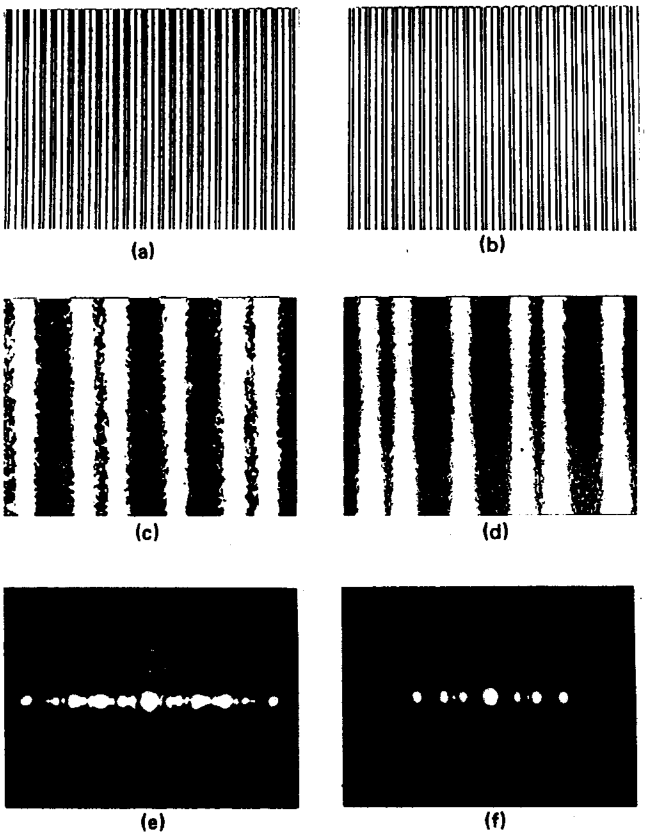

Fig. 3. A comparison of the gratings and resulting diffraction pattern for Polaroid HC Instant Copy Film (left column) and Kodak Technical Pan Film (right column). Here, (a) and (b) are 30× negative enlargements of the slides used to produce the diffraction patterns shown in (e) and (f), respectively; (c) and (d) are reproduced from 100× photomicrographs showing the relative grain size of the two films used. The horizontal dark streak in 3(d) is due to uneven lighting during reproduction, and not present on the original slide.

film provides less transmission of the laser light used to generate the pattern, and the intensities of the secondary maxima at 2.65 and 4.0 are unchanged in relative value even though the intensities relative to the central maxima are reduced.

IV. CONCLUSIONS

(1) A computer with a high-resolution monitor has been successfully used to generate one-dimensional quasi-crystal gratings that when photographed with an appropriate reduction will produce an acceptable diffraction pattern.

(2) The algebraic theory for location and intensity of diffraction maxima has been compared to an experimentally produced pattern. Agreement is generally good for location of diffraction maxima, but poor in most cases for actual relative intensity measurements.

(3) At the current time, better results for photographic production of gratings remains with conventional high-contrast negative emulsions. However, while either film will produce a satisfactory result for locating diffraction maxima, the polaroid film has a distinct advantage in processing speed.

(4) A rapid and easily reproducible method for producing custom gratings for a variety of instructional uses has been described. This work could easily be extended to two-dimensional patterns within the limitations of the pixel array of the computer monitor.

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